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Rayleigh–Benard convection in rotating fluids

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Abstract

Linear and weakly nonlinear properties of Rayleigh–Benard convection in rotating fluids are investigated. Linear stability analysis is studied to investigate analytically the effect of Coriolis force on gravity-driven convection for idealised stress-free boundary conditions. We have derived a nonlinear one-dimensional Landau–Ginzburg equation with real coefficients near the onset of stationary convection at the supercritical pitchfork bifurcation. A coupled Landau–Ginzburg type equations with complex coefficients near the onset of oscillatory convection at the supercritical Hopf bifurcation are derived and discussed the stability regions of travelling and standing waves.

Keywords: Stationary and oscillatory convection; Coriolis force; Bifurcation points; Landau–Ginzburg equation; Travelling and standing wave convection

1. Introduction

Rayleigh–Benard convection with rotation about a vertical axis is an interesting hydrodynamic system since it combines the elements of thermal buoyancy and rotation induced Coriolis and centrifugal forces. This is a simple model that contains the fundamental forces that control the atmospheric and oceanic circulation. The importance of Rayleigh–Benard convection with rotation in atmospheric and oceanic flow has enviced a significant theoretical and experimental interest in the problem [2,3,1,10]. Addition of Coriolis forces on the Rayleigh–Benard convection induces another control parameter into the problem, namely the Taylor number which is a measure of rotation rate. The multiplicity of control parameters makes this system an interesting one for the study of hydrodynamic stability, bifurcation and turbulence [7]. Chandrasekhar [4] derived the critical Rayleigh number as a function of Taylor number. For theoretical simplicity, he considered an infinite layer of fluid. But in many practical situations as in a laboratory experiment, there are vertical boundaries on side walls and the horizontal dimension of the convection cells is comparable to their vertical depths. It was also found that the experimental observations were not in line with theoretical predictions made by Chandrasekhar. Davies-Jones and Oilman [5] found that for certain aspect rations (width to depth ratios) and large enough Taylor number, the critical Rayleigh number for steady convection is less than that for infinite case even though the system is more constrained. Khiri [6] has considered the problem of Coriolis effect on convection for a low Prandtl number for stress-free boundary conditions (even though he mentions as rigid–rigid boundary condition). In the next section, we have written the basic equations which describe the Rayleigh–Benard convection with rotation. In Section 3, we revisit Chandrasekhar [4] and carry out a linear stability analysis and identify certain parameter regimes for the onset of stationary and oscillatory convection. In Section 4, we derive an amplitude equation, which is Landau–Ginzburg equation with real coefficients, near the onset of stationary convection at supercritical pitchfork bifurcation [8]. In Section 5, we derive two nonlinear one-dimensional time-dependent coupled Landau–Ginzburg type equation in complex amplitudes $A_{1R}(X,\tau,T), A_{1L}(X,\tau,T)$ with complex coefficients near the onset of oscillatory convection at supercritical

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Hopf bifurcation. Here $A_{1R}(X,\tau,T)$, $A_{1L}(X,\tau,T)$ stands for amplitudes of right hand and left hand travelling waves. Following Matthews and Rucklidge [9], we have dropped slow space dependence and obtained two ODE's in $A_{1R}(T)$ and $A_{1L}(T)$ with complex coefficients and discussed the stability regions of travelling and standing waves. In Section 6, we write conclusions of our paper.

2. Basic equations

Let us consider an infinite horizontal layer of fluid which is kept rotating at a constant angular velocity $\widehat{\Omega} = \Omega\hat{e}_z$ and is also heated from below. We use a cartesian system of co-ordinates whose dimensionless horizontal co-ordinates x, y and vertical co-ordinate z are scaled on d, the depth of the fluid layer. The velocity $\vec{V}(u,v,w)$, the temperature θ, time t and pressure p are nondimensionalised by $\frac{\kappa}{d}$, βd, $\frac{d^2}{\kappa}$ and $\rho_0\kappa^2 d^{-2}$. Here κ is thermal diffusivity, v is viscosity, β is adverse temperature gradient and ρ_0 is the density. The dimensionless equations for a rotating Rayleigh–Benard system in the Oberbeck–Boussinesq approximation are:

$$\nabla\cdot\vec{V} = 0, \tag{1}$$

$$\frac{1}{Pr}\left[\frac{\partial\vec{V}}{\partial t} + (\vec{V}\cdot\nabla)\vec{V}\right] = -\nabla\left(p - \frac{TaPr}{8}|\hat{e}_z\times\vec{r}|\right) + \nabla^2\vec{V} + R\theta\hat{e}_z + Ta^{\frac{1}{2}}(\vec{V}\times\hat{e}_z), \tag{2}$$

$$\frac{\partial\theta}{\partial t} + (\vec{V}\cdot\nabla)\theta = w + \nabla^2\theta, \tag{3}$$

where $\hat{e}_z$ is the unit vector along the axis of rotation. The dimensionless numbers required for the description of the motion are: Rayleigh number $R = \frac{g\alpha\beta d^4}{\kappa v}$, Prandtl number $Pr = \frac{v}{\kappa}$ and Taylor number $Ta = \frac{4\Omega^2 d^4}{v^2}$. Now it is convenient to reduce the basic Eqs. (1)–(3) into a single equation. To do this, we take curl of Eq. (2) and obtain

$$\left(\frac{1}{Pr}\frac{\partial}{\partial t} - \nabla^2\right)\vec{\omega} - R\nabla\times(\theta\hat{e}_z) - Ta^{\frac{1}{2}}\nabla\times\left(\vec{V}\times\hat{e}_z\right) = -\frac{1}{Pr}\nabla\times[(\vec{V}\cdot\nabla)\vec{V}], \tag{4}$$

where $\vec{\omega} = \nabla\times\vec{V}$ is the vorticity and $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$ The curl of Eq. (4), in turn, after use of Eq. (1) gives

$$\left(\frac{1}{Pr}\frac{\partial}{\partial t} - \nabla^2\right)\nabla^2\vec{V} - R\left[\nabla^2(\theta\hat{e}_z) - \nabla\left(\frac{\partial\theta}{\partial z}\right)\right] + Ta^{\frac{1}{2}}\frac{\partial\vec{\omega}}{\partial z} = \frac{1}{Pr}\{\nabla\times[\nabla\times(\vec{V}\cdot\nabla)\vec{V}]\}. \tag{5}$$

The z-component of Eqs. (4) and (5) are

$$\left(\frac{1}{Pr}\frac{\partial}{\partial t} - \nabla^2\right)\omega_z - Ta^{\frac{1}{2}}\frac{\partial w}{\partial z} = -\frac{1}{Pr}\left[(\vec{V}\cdot\nabla)\omega_z - (\vec{\omega}\cdot\nabla)w\right], \tag{6}$$

$$\left(\frac{1}{Pr}\frac{\partial}{\partial t} - \nabla^2\right)\nabla^2 w - R\nabla_h^2\theta + Ta^{\frac{1}{2}}\frac{\partial\omega_z}{\partial z} = \frac{1}{Pr}\hat{e}_z\cdot\{\nabla\times\left[(\vec{V}\cdot\nabla)\vec{\omega}\right] - \nabla\times[(\vec{\omega}\cdot\nabla)\vec{V}]\}, \tag{7}$$

where ω_z and w are the z-components of vorticity and velocity, respectively and $\nabla_h^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ is a horizontal Laplacian operator. Eliminating θ and ω_z from the linear part of Eqs. (3), (6) and (7), we get

$$\mathscr{L}w = \mathscr{N}, \tag{8}$$

where

$$\mathscr{L} = \left(\frac{1}{Pr}\frac{\partial}{\partial t} - \nabla^2\right)^2\left(\frac{\partial}{\partial t} - \nabla^2\right)\nabla^2 + Ta\left(\frac{\partial}{\partial t} - \nabla^2\right)\frac{\partial^2}{\partial z^2} - R\nabla_h^2\left(\frac{1}{Pr}\frac{\partial}{\partial t} - \nabla^2\right), \tag{9}$$

and

$$\begin{aligned}\mathscr{N} = & -R\left(\frac{1}{Pr}\frac{\partial}{\partial t} - \nabla^2\right)\nabla_h^2(\vec{V}.\nabla)\theta \\ & + \frac{Ta^{\frac{1}{2}}}{Pr}\left(\frac{\partial}{\partial t} - \nabla^2\right)\frac{\partial}{\partial z}\left[(\vec{V}\cdot\nabla)\omega_z - (\vec{\omega}\cdot\nabla)w\right] \\ & - \frac{1}{Pr}\left(\frac{1}{Pr}\frac{\partial}{\partial t} - \nabla^2\right)\left(\frac{\partial}{\partial t} - \nabla^2\right)\hat{e}_z\cdot\nabla\times\left[(\vec{V}\cdot\nabla)\vec{\omega}\right] \\ & + \frac{1}{Pr}\left(\frac{1}{Pr}\frac{\partial}{\partial t} - \nabla^2\right)\left(\frac{\partial}{\partial t} - \nabla^2\right)\hat{e}_z\cdot\nabla\times[(\vec{\omega}\cdot\nabla)\vec{V}].\end{aligned} \tag{10}$$

Boundary conditions:

The fluid is confined between the planes $z = 0, z = 1$ and is rotating about z-axis. For free–free (stress-free) boundary conditions, we have

$$\theta = 0, \quad w = 0, \quad \frac{\partial w_z}{\partial z} = 0 \quad \text{and} \quad \frac{\partial^2 w}{\partial z^2} = 0,$$
$$\text{on } z = 0, \ z = 1 \quad \text{for all } x, y,$$

and for rigid–rigid boundary conditions, we have

$$\theta = 0, \quad w = 0, \quad w_z = 0 \quad \text{and} \quad \frac{\partial w}{\partial z} = 0$$
$$\text{on } z = 0, \ z = 1 \quad \text{for all } x, y.$$

In our paper, we use stress-free boundary conditions.

3. Linear stability analysis

We perform the linear stability analysis of the problem by substituting

$$w = W(z)e^{iq_x x + pt}, \tag{11}$$

into linearized version of Eq. (8) viz.

$$\mathscr{L}w = 0,$$

and obtaining an equation

$$\left[(D^2 - q^2)(D^2 - q^2 - p)\left(D^2 - q^2 - \frac{p}{Pr}\right)^2 + TaD^2(D^2 - q^2 - p) + Rq^2\left(D^2 - q^2 - \frac{p}{Pr}\right)\right]W(z) = 0, \tag{12}$$

where

$$q^2 = q_x^2 \quad \text{and } D = \frac{\mathrm{d}}{\mathrm{d}z}.$$

We consider only idealized boundary conditions (free–free boundary conditions). Hence, W and all its even derivatives vanish at $z = 0$ and $z = 1$. Substituting $W(z) = \sin \pi z$ and $p = \mathrm{i}\omega$ in Eq. (12), we get

$$R = \frac{1}{q^2}[A_1 + \mathrm{i}\omega(A_2\omega^2 + A_3)], \tag{13}$$

where

$$A_1 = \delta^2\left(\delta^4 - \frac{\omega^2}{Pr}\right) + \frac{Ta\pi^2\left(\delta^4 + \frac{\omega^2}{Pr}\right)}{\delta^4 + \frac{\omega^2}{Pr^2}}, \tag{14a}$$

$$A_2 = \frac{\delta^2}{Pr^2}\left(1 + \frac{1}{Pr}\right), \tag{14b}$$

$$A_3 = \delta^6\left(1 + \frac{1}{Pr}\right) + Ta\pi^2\left(1 - \frac{1}{Pr}\right), \tag{14c}$$

and $\delta^2 = \pi^2 + q^2$. From relation Eq. (14b), $A_2 > 0$.

(a) Stationary convection ($\omega = 0$):

Substituting $\omega = 0$ in Eq. (13), we get

$$R_{\mathrm{s}} = \frac{\delta_{\mathrm{s}}^6 + Ta\pi^2}{q_{\mathrm{s}}^2}, \tag{15}$$

where $\delta_{\mathrm{s}}^2 = \pi^2 + q_{\mathrm{s}}^2$. Here R_{s} is the value of the Rayleigh number for stationary convection. The critical value of R_{s} is obtained for $q = q_{\mathrm{sc}}$, where

$$2\left(\frac{q_{\mathrm{sc}}}{\pi}\right)^6 + 3\left(\frac{q_{\mathrm{sc}}}{\pi}\right)^4 = 1 + \frac{Ta}{\pi^4}. \tag{16}$$

Threshold for the onset of stationary convection is given by Eq. (15), with $q = q_{\mathrm{sc}}$. Thus

$$R_{\mathrm{sc}} = \frac{\delta_{\mathrm{sc}}^6 + Ta\pi^2}{q_{\mathrm{sc}}^2}, \tag{17}$$

where $\delta_{\mathrm{sc}}^2 = \pi^2 + q_{\mathrm{sc}}^2$. For $\frac{Ta}{\pi^4} \gg 1$ (for large Taylor number), the required root of Eq. (16) becomes

$$\left(\frac{q_{\mathrm{sc}}}{\pi}\right) \simeq \left(\frac{Ta}{2\pi^4}\right)^{\frac{1}{6}}.$$

The corresponding asymptotic values of q_{sc} and R_{sc} are

$$q_{\mathrm{sc}} \simeq \left(\frac{\pi^2 Ta}{2}\right)^{\frac{1}{6}}, \tag{18a}$$

$$R_{\mathrm{sc}} \simeq 3\pi^4\left(\frac{Ta}{2\pi^4}\right)^{\frac{2}{3}}. \tag{18b}$$

In the free–free boundary conditions we have proved that for large Taylor number

$$R_{\mathrm{sc}} \propto Ta^{\frac{2}{3}} \quad \text{and } q_{\mathrm{sc}} \propto Ta^{\frac{1}{6}}. \tag{19}$$

This is also true for rigid–rigid and rigid–free boundary conditions.

(b) Oscillatory convection ($\omega^2 > 0$):

For the oscillatory convection ($\omega \neq 0$) and from Eq. (13), R will be complex. But the physical meaning of R requires it to be real. The condition that R is real implies that imaginary part of Eq. (13) is zero, i.e.,

$$A_2\omega^2 + A_3 = 0, \tag{20}$$

where A_2, A_3 are given by Eqs. (14b) and (14c). For oscillatory convection $\omega^2 = -\frac{A_3}{A_2} > 0$, i.e.,

$$\omega^2 = \frac{Pr^2}{\delta_0^2(1 + Pr)}[Ta\pi^2(1 - Pr) - \delta_0^6(1 + Pr)], \tag{21}$$

where $\delta_0^2 = \pi^2 + q_0^2$. Substituting ω^2 from Eq. (21) into the real part of Eq. (13), we get

$$R_0 = \frac{2(1 + Pr)}{q^2}\left[\delta_0^6 + \frac{Ta\pi^2 Pr^2}{(1 + Pr)^2}\right]. \tag{22}$$

A necessary condition for $\omega^2 > 0$ is $Pr < 1$. However, this is not sufficient condition and one must have in addition

$$\begin{aligned} &Ta > \frac{(1 + Pr)\delta_0^6}{\pi^2(1 - Pr)} \\ &Ta = Ta_{\mathrm{c}} = \frac{\delta_{\mathrm{c}}^6(1 + Pr)}{\pi^2(1 - Pr)}, \ q = q_c \end{aligned} \tag{23}$$

is a solution of $A_3(Ta_{\mathrm{c}}) = 0$ and corresponds to a Takens–Bogdanov bifurcation point. At Takens–Bogdanov bifurcation point $q_0 = q_{\mathrm{s}} = q_{\mathrm{c}}$ and $A_3(q_{\mathrm{c}}) = 0$. We note from real part of Eq. (22) that if $\omega^2 > 0$ then $R_0(q_0)$ will be less than $R_s(q_0)$ and not $R_{\mathrm{s}}(q_{\mathrm{s}})$ given by Eq. (15), which corresponds to onset of stationary convection. However, at Takens–Bogdanov bifurcation point

$$R_0(q_0) = R_{\mathrm{s}}(q_{\mathrm{s}}) = R_{\mathrm{c}}(q_{\mathrm{c}}), \quad q_0 = q_{\mathrm{s}} = q_{\mathrm{c}}$$

and $\omega^2 = 0$ is a double zero at $Ta = Ta_{\mathrm{c}}(q_{\mathrm{c}})$. The Takens–Bogdanov bifurcation point occurs where neutral curves for Hopf and pitchfork bifurcation meet and only a single wave number is present viz. $q_0 = q_{\mathrm{s}} = q_{\mathrm{c}}$. If $q_{\mathrm{c}} > q_{\mathrm{sc}}$ then for all $q < q_{\mathrm{c}}$ the first instability to set in is an oscillatory instability.

The critical wave number corresponding to the onset of oscillatory convection for given parameters Pr and Ta is obtained for $q = q_{\mathrm{oc}}$ from the following equation

$$2\left(\frac{q_{\mathrm{oc}}}{\pi}\right)^6 + 3\left(\frac{q_{\mathrm{oc}}}{\pi}\right)^4 = 1 + \frac{TaPr^2}{\pi^4(1 + Pr)^2}. \tag{24}$$

For large Taylor number, the required root of Eq. (24) becomes

$$\frac{q_{\mathrm{oc}}}{\pi} = \left(\frac{TaPr^2}{2\pi^4(1 + Pr)^2}\right)^{\frac{1}{6}}.$$

The corresponding asymptotic behavior of q_{oc} and R_{oc} for large Taylor number are (Chandrasekhar [4])

$$q_{oc} \simeq \left(\frac{TaPr^2\pi^2}{2(1+Pr)^2}\right)^{\frac{1}{6}}, \tag{25a}$$

$$R_{oc} \simeq 2\pi^4(1+Pr)\left[3\left(\frac{TaPr^2}{2\pi^4(1+Pr)^2}\right)^{\frac{2}{3}}\right]. \tag{25b}$$

From Eqs. (18b) and (25b), $R_{oc} \to R_{sc}$ as $Ta \to \infty$ implies that for large Taylor number

$$\frac{2Pr^{\frac{4}{3}}}{(1+Pr)^{\frac{1}{3}}} = 1. \tag{26}$$

Root of Eq. (26) is (Chandrasekhar [4]) $Pr = Pr_c = 0.67659$. Thus $R_{oc}(q_{oc}) \to R_{sc}(q_{sc})$ at $Pr = Pr_c$. From the monotonic dependence of q_{oc} and q_{sc} on Ta, we may conclude that for $Pr > Pr_c, R_{oc} > R_{sc}$ for all Ta. Hence for $1 > Pr > Pr_c$, instability will always manifest itself, first as stationary convection. For $Pr < Pr_c$, there exist a $Ta(Pr)$ such that for $Ta \leqslant Ta(Pr)$ the onset of instability will be stationary convection at pitchfork bifurcation while for $Ta > Ta(Pr)$ it will be oscillatory convection at Hopf bifurcation. $Ta(Pr)$ is a function of Prandtl number Pr and for $Ta = Ta(Pr)$

$$R_{ct}(q_{oc}) = R_{sc}(q_{sc}) \quad \text{but } q_{oc} \neq q_{sc}. \tag{27}$$

This point is known as codimension two bifurcation point and is intersection between a Hopf and pitchfork bifurcation with distinct wave numbers. Thus Takens–Bogdanov bifurcation point and codimension two bifurcation point are different. There is no simple formula to give $Ta(Pr)$ as a function of Pr.

In Figs. 1a–d, solid line represent stationary convection (pitchfork bifurcation) and dotted line denotes oscillatory convection (Hopf bifurcation) which are plotted in (q, R)-plane. The value of ω^2 decreases on dotted line when q increases and ω^2 takes zero value at the intersection of solid and dotted line.

In Figs. 1a–d, we have shown the effect of Taylor number Ta, over the onset of both stationary and oscillatory convection. From these figures we can say that when Ta increases, then the onset of both stationary and oscillatory convection will increase. This implies that rotation rate

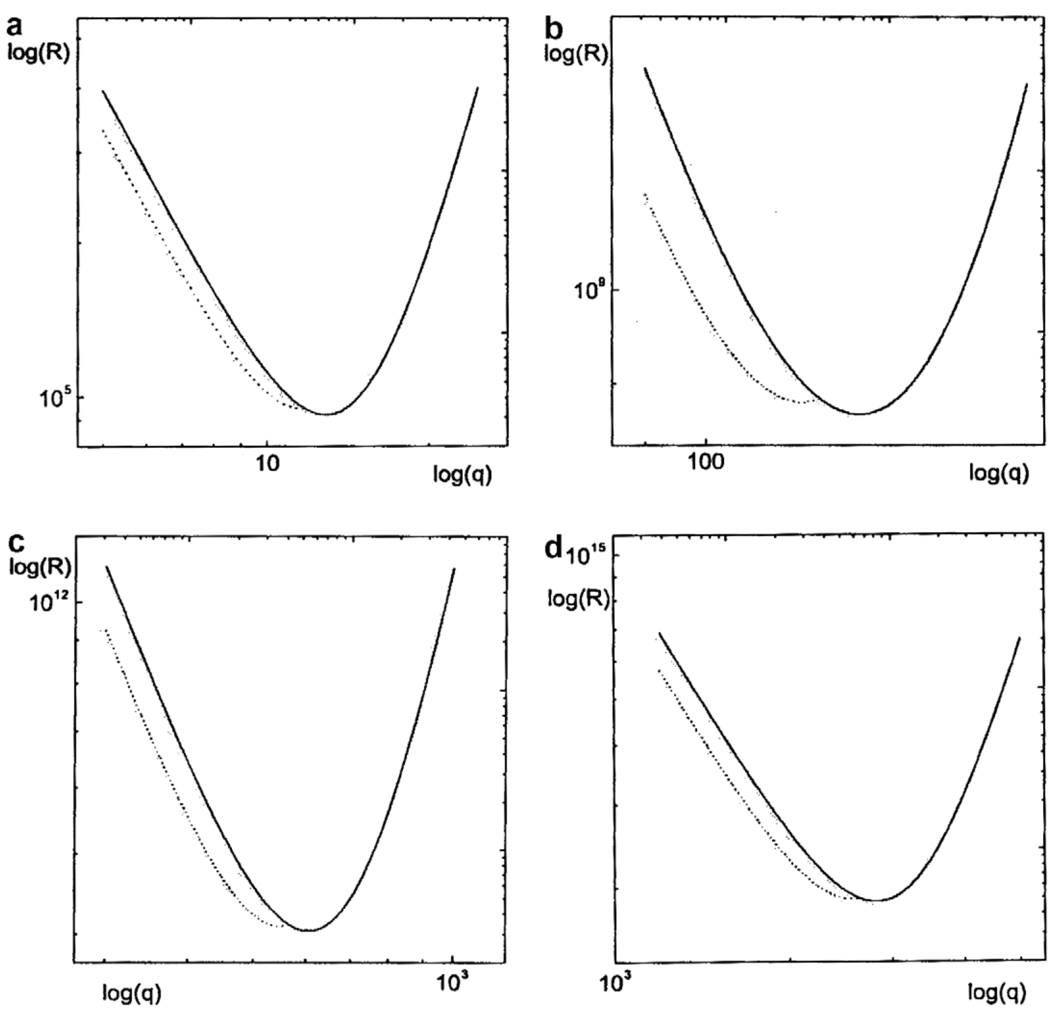

Fig. 1. Marginal stability curves (stationary convection-solid lines, oscillatory convection-dotted lines) are plotted for $Pr = 0.5$, and (a) $Ta = 10^6$, (b) $Ta = 10^{12}$, (c) $Ta = 10^{16}$, (d) $Ta = 10^{20}$.

inhibits the onset of convection. This result is true for other parameter Pr also. In Figs. 1a–d, we can see three types of bifurcations like pitchfork bifurcation, Hopf bifurcation, Takens–Bogdanov bifurcation point, (the intersection point of solid and dotted line).

4. Two-dimensional Landau–Ginzburg equation at the onset of stationary convection

The existence of threshold and the cellular structure (critical wave number) for a fixed Taylor number Ta are main characteristics of the stationary convection in a rotating fluid. Here, we consider the region at supercritical pitchfork bifurcation ($R > R_{sc}$). We write the solution of Eqs. (1)–(3) in the power series of ϵ given as follows

$$f = \epsilon f_0 + \epsilon^2 f_1 + \epsilon^3 f_2 + \cdots, \tag{28}$$

where

$$f = f(u, v, w, \omega_x, \omega_y, \omega_z, \theta)$$

with the first approximation given by the eigenvector of the linearized problem:

$$\begin{aligned}
u_0 &= \frac{i\pi}{q_{sc}}[A(X,Y,T)e^{iq_{sc}x}\cos\pi z - \text{c.c.}],\\
v_0 &= \frac{-i\pi Ta^{\frac{1}{2}}}{q_{sc}\delta_{sc}^2}[A(X,Y,T)e^{iq_{sc}x}\cos\pi z - \text{c.c.}],\\
w_0 &= A(X,Y,T)e^{iq_{sc}x}\sin\pi z + \text{c.c.},\\
\omega_{x_0} &= \frac{-i\pi^2 Ta^{\frac{1}{2}}}{q_{sc}\delta_{sc}^2}\left[A(X,Y,T)e^{iq_{sc}x}\sin\pi z - \text{c.c.}\right],\\
\omega_{y_0} &= \frac{-i\delta_{sc}^2}{q_{sc}}\left[A(X,Y,T)e^{iq_{sc}x}\sin\pi z - \text{c.c.}\right],\\
\omega_{z_0} &= \frac{\pi Ta^{\frac{1}{2}}}{\delta_{sc}^2}[A(X,Y,T)e^{iq_{sc}x}\cos\pi z + \text{c.c.}],\\
\theta_0 &= \frac{1}{\delta_{sc}^2}[A(X,Y,T)e^{iq_{sc}x}\sin\pi z + \text{c.c.}].
\end{aligned} \tag{29}$$

The amplitude $A(X,Y,T)$ is allowed to depend on slow space and time variables

$$X = \epsilon x, \quad Y = \epsilon^{\frac{1}{2}}y, \quad T = \epsilon^2 t, \quad z = z. \tag{30}$$

We expand the linear operator $\mathscr{L}$ and nonlinear operator $\mathscr{N}$ as the following power series

$$\mathscr{L} = \mathscr{L}_0 + \epsilon\mathscr{L}_1 + \epsilon^2\mathscr{L}_2 + \cdots, \tag{31a}$$

$$\mathscr{N} = \epsilon^2\mathscr{N}_0 + \epsilon^3\mathscr{N}_1 + \cdots \tag{31b}$$

Substituting Eqs. (31a), (31b) and (28) into Eq. (8), we get by equating the coefficients of $\epsilon, \epsilon^2, \epsilon^3$

$$\mathscr{L}_0 w_0 = 0, \tag{32a}$$

$$\mathscr{L}_0 w_1 + \mathscr{L}_1 w_0 = \mathscr{N}_0, \tag{32b}$$

$$\mathscr{L}_0 w_2 + \mathscr{L}_1 w_1 + \mathscr{L}_2 w_0 = \mathscr{N}_1, \tag{32c}$$

where

$$\mathscr{L}_0 = -\nabla^2\left(\nabla^6 + Ta\frac{\partial^2}{\partial z^2} - R_{sc}\frac{\partial^2}{\partial x^2}\right), \tag{33a}$$

$$\mathscr{L}_1 = -2\left(\frac{\partial^2}{\partial x\partial X} + \frac{1}{2}\frac{\partial^2}{\partial Y^2}\right)\left[4\nabla^6 + Ta\frac{\partial^2}{\partial z^2} - R_{sc}\left(2\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2}\right)\right], \tag{33b}$$

$$\begin{aligned}
\mathscr{L}_2 = {} & \frac{\partial}{\partial T}\left[\left(1 + \frac{2}{\mathrm{Pr}}\right)\nabla^6 + Ta\frac{\partial^2}{\partial z^2} - \frac{R_{sc}}{Pr}\frac{\partial^2}{\partial x^2}\right]\\
& - \frac{\partial^2}{\partial X^2}\left[4\nabla^6 + Ta\frac{\partial^2}{\partial z^2} - R_{sc}\left(2\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2}\right)\right]\\
& + \left(\frac{\partial^2}{\partial x\partial X} + \frac{1}{2}\frac{\partial^2}{\partial Y^2}\right)^2(-24\nabla^4 + 4R_{sc}) + R_{sc}\frac{\partial^2}{\partial x^2}\nabla^2.
\end{aligned} \tag{33c}$$

Eq. (32a) gives the critical Rayleigh number for the onset of stationary convection

$$R_{sc} = \frac{\delta_{sc}^6 + Ta\pi^2}{q_{sc}^2}.$$

Here q_{sc} is given by Eq. (16), this implies that

$$\mathscr{L}_1 w_0 = 0.$$

Hence Eq. (32b) becomes

$$\mathscr{L}_0 w_1 = \mathscr{N}_0.$$

Substituting the zeroth order solutions Eq. (29) into $\mathscr{N}_0$, we find that $\mathscr{N}_0 = 0$, hence $w_1 = 0$. From equation of continuity we find that $u_1 = 0$. The relevant equations for ω_{z_1} and θ_1, respectively are

$$\left(\frac{1}{Pr}\frac{\partial}{\partial t} - \nabla^2\right)\omega_{z_x} = Ta^{\frac{1}{2}}\frac{\partial w_1}{\partial z} - \frac{1}{Pr}\left[(\vec{V}_0\cdot\nabla)\omega_{z_0} - (\vec{\omega}_0\cdot\nabla)w_0\right], \tag{34a}$$

$$\left(\frac{\partial}{\partial t} - \nabla^2\right)\theta_1 - w_1 - (\vec{V}_0\cdot\nabla)\theta_0. \tag{34b}$$

Substituting zeroth order approximations from Eq. (30) into Eqs. (34a) and (34b) and using $w_1 = 0$, we get

$$\begin{aligned}
\omega_{z_1} &= \frac{Ta^{\frac{1}{2}}\pi^2}{2Prq_{sc}^2\delta_{sc}^2}[A^2 e^{2iq_{sc}x} + \text{c.c.}],\\
\omega_{x_1} &= \omega_{y_1} = 0,\\
\theta_1 &= \frac{-1}{2\pi\delta_{sc}^2}|A|^2\sin 2\pi z,\\
v_1 &= \frac{-iTa^{\frac{1}{2}}\pi^2}{4Prq_{sc}^3\delta_{sc}^2}[A^2 e^{2iq_{sc}x} - \text{c.c.}].
\end{aligned} \tag{35}$$

The solvability criterion of Eq. (32c) gives us the amplitude equation

$$\lambda_0\frac{\partial A}{\partial T} - \lambda_1\left(\frac{\partial}{\partial X} - \frac{i}{2q_{sc}}\frac{\partial^2}{\partial Y^2}\right)^2 A - \lambda_2 A + \lambda_3|A|^2 A = 0, \tag{36}$$

where

$$\begin{aligned}\lambda_0 &= \left(1+\frac{1}{Pr}\right)\delta_{sc}^6 + \left(1-\frac{1}{Pr}\right)\pi^2 Ta,\\ \lambda_1 &= 4\left[\delta_{sc}^4\left(5q_{sc}^2-\pi^2\right) - Ta\pi^2\right],\\ \lambda_2 &= R_{sc}q_{sc}^2\delta_{sc}^2,\\ \lambda_3 &= \frac{R_{sc}q_{sc}^2}{2} - \frac{Ta^{\frac{1}{2}}\pi^4}{Prq_{sc}^2}.\end{aligned} \tag{37}$$

Eq. (36) is time dependent non-linear two-dimensional Landau–Ginzburg equation describing the effect of rotation on Rayleigh–Benard problem. We notice that λ_1, λ_2 are always positive and λ_3 is positive if

$$R_{sc} > \frac{2\pi^4 Ta^{\frac{1}{2}}}{Prq_{sc}^4}. \tag{38}$$

At $\lambda_3 = 0$, we get tricritical bifurcation point. The pitchfork bifurcation is supercritical if $\lambda_3 > 0$ and subcritical if $\lambda_3 < 0$ (see Fig. 2). $\lambda_0 = 0$ if $Ta = Ta_c$ and $\lambda_0 > 0$ if $Ta < Ta_c$. Dropping time dependence and Y-dependence from Eq. (36), we get

$$\frac{d^2A}{dX^2} + \frac{\lambda_2}{\lambda_1}A\left(1-\frac{\lambda_3}{\lambda_2}|A|^2\right) = 0. \tag{39}$$

Here, we assume that, condition Eq. (38) is satisfied so that λ_2 and λ_3 are always positive. Since $\lambda_1 > 0$, the solution of the Eq. (39) is given by

$$A(X) = A_0 \tanh\left(\frac{X}{\Lambda}\right),$$

where

$$A_0 = \sqrt{\frac{\lambda_2}{\lambda_3}} \quad \text{and} \quad \Lambda = \sqrt{\frac{2\lambda_1}{\lambda_2}}. \tag{40}$$

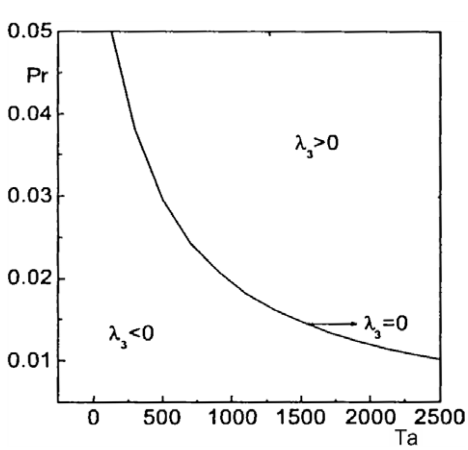

Fig. 2. In $\lambda_3 < 0$ region, the pitchfork bifurcation is subcritical and in $\lambda_3 > 0$ region the pitchfork bifurcation is supercritical. $\lambda_3 = 0$ gives tricritical bifurcation point.

4.1. Long wave-length instabilities

A secondary instability arises from a neutral mode associated with a symmetry of the governing equations broken by the primary instabilities.

In order to study the properties of a structure with a given phase winding number $\delta q_s = q - q_{sc}$, we substitute

$$A(X,Y,T) = \widetilde{A}(X,Y,T)e^{i\delta q_s X} \text{ (stationary solutions)},$$

into Eq. (36) and we obtain

$$\begin{aligned}\lambda_0\frac{\partial\widetilde{A}}{\partial T} &= (\lambda_2 - \lambda_1\delta q_s^2)\widetilde{A} + 2i\lambda_1\delta q_s\left(\frac{\partial}{\partial X} - \frac{i}{2q_{sc}}\frac{\partial^2}{\partial Y^2}\right)\widetilde{A}\\ &\quad + \lambda_1\left(\frac{\partial}{\partial X} - \frac{i}{2q_{sc}}\frac{\partial^2}{\partial Y^2}\right)^2\widetilde{A} - \lambda_3|\widetilde{A}|^2\widetilde{A}\\ &= 0.\end{aligned} \tag{41}$$

The steady state uniform solution of Eq. (41) is

$$\widetilde{A} = \widetilde{A}_0 = \left[\frac{(\lambda_2 - \lambda_1\delta q_s^2)}{\lambda_3}\right]^{\frac{1}{2}}. \tag{42}$$

Let $\tilde{u}(X,Y,T) + i\tilde{v}(X,Y,T)$ be an infinitesimal perturbation from a uniform steady state solution $\widetilde{A}_0$ given by Eq. (42). Now substituting

$$\widetilde{A} = \left[\frac{(\lambda_2 - \lambda_1\delta q_s^2)}{\lambda_3}\right] + \tilde{u} + i\tilde{v}$$

into Eq. (41) and equating real and imaginary parts, we obtain

$$\begin{aligned}\lambda_0\frac{\partial\tilde{u}}{\partial T} &= \left[-2(\lambda_2 - \lambda_1\delta q_s^2) + \lambda_1\frac{\partial^2}{\partial X^2} + \frac{\lambda_1\delta q_s}{q_{sc}}\frac{\partial^2}{\partial Y^2} - \frac{\lambda_1}{4q_{sc}^2}\frac{\partial^4}{\partial Y^4}\right]\tilde{u}\\ &\quad - \left(2\lambda_1\delta q_s - \frac{\lambda_1}{q_{sc}}\frac{\partial^2}{\partial Y^2}\right)\frac{\partial\tilde{v}}{\partial X},\end{aligned} \tag{43a}$$

$$\begin{aligned}\lambda_0\frac{\partial\tilde{v}}{\partial T} &= \left(2\lambda_1\delta q_s - \frac{\lambda_1}{q_{sc}}\frac{\partial^2}{\partial Y^2}\right)\frac{\partial\tilde{u}}{\partial X}\\ &\quad + \lambda_1\left(\frac{\partial^2}{\partial X^2} + \frac{\delta q_s}{q_{sc}}\frac{\partial^2}{\partial Y^2} - \frac{1}{4q_{sc}^2}\frac{\partial^4}{\partial Y^4}\right)\tilde{v}.\end{aligned} \tag{43b}$$

We analyze Eqs. (43a) and (43b) by using normal modes of the form

$$\begin{aligned}\tilde{u} &= Ue^{(ST)}\cos(q_X X)\cos(q_Y Y),\\ \tilde{v} &= Ve^{(ST)}\sin(q_X X)\cos(q_Y Y).\end{aligned} \tag{44}$$

Putting solutions Eq. (44) into Eqs. (43a) and (43b) we get,

$$\left[\lambda_0 S + 2(\lambda_2 - \lambda_1\delta q_s^2) + \lambda_1 q_X^2 + \frac{\lambda_1\delta q_s}{q_{sc}}q_Y^2 + \frac{\lambda_1}{4q_{sc}^2}q_Y^4\right]U + \left(2\delta q_s + \frac{q_Y^2}{q_{sc}}\right)\lambda_1 q_X V = 0, \tag{45a}$$

$$\lambda_1 q_X\left(2\delta q_s + \frac{q_Y^2}{q_{sc}}\right)U + \left(\lambda_0 S + \lambda_1 q_X^2 + \frac{\lambda_1\delta q_s}{q_{sc}}q_Y^2 + \frac{\lambda_1}{4q_{sc}^2}q_Y^4\right)V = 0. \tag{45b}$$

On solving Eqs. (45a) and (45b) we get,

$$\lambda_0^2 S^2 + 2S\left[2\lambda_0(\lambda_2 - \lambda_1\delta q_s^2) + \lambda_0\lambda_1 q_X^2 + \frac{\lambda_0\lambda_1}{q_{sc}} q_Y^2 \delta q_s + \frac{\lambda_0\lambda_1}{4q_{sc}^2} q_Y^4\right] + \left[2(\lambda_2 - \lambda_1\delta q_s^2) + \lambda_1 q_X^2 + \frac{\lambda_1}{q_{sc}} q_Y^2 \delta q_s + \frac{\lambda_1}{4q_{sc}^2} q_Y^4\right] \times \left[\lambda_1 q_X^2 + \frac{\lambda_2 \delta q_s}{q_{sc}} q_Y^2 + \frac{\lambda_1}{4q_{sc}^2} q_Y^4\right] - q_X^2\left(2\lambda_1\delta q_s + \frac{\lambda_1}{q_{sc}} q_Y^2\right)^2 = 0, \tag{46}$$

whose roots $(S\pm)$ are real. Here $(S\pm)$ defined as

$$S(\pm) = -\frac{1}{\lambda_0^2}\left\{\left[2\lambda_0(\lambda_2 - \lambda_1\delta q_s^2) + \lambda_0\lambda_1 q_X^2 + \frac{\lambda_0\lambda_1}{q_{sc}} q_Y^2\delta q_s + \frac{\lambda_0\lambda_1}{4q_{sc}^2} q_Y^4\right] \pm \left[(2\lambda_0(\lambda_2 - \lambda_1\delta q_s^2))^2 + \lambda_1^2 q_X^2\left(2\delta q_s + \frac{q_Y^2}{q_{sc}}\right)^2\right]^{\frac{1}{2}}\right\}, \tag{47}$$

solution $S(-)$ is clearly negative, thus the corresponding mode is stable and if $S(+)$ is positive then rolls can be unstable. Symmetry considerations help us to restrict the study of $S(+)$ to a domain $(q_X \geqslant 0, q_Y \geqslant 0)$.

(a) Longitudinal perturbations and Eckhaus instability: Inserting $q_Y = 0$ into Eq. (46); we get

$$\lambda_0^2 S^2 + 2S(2\lambda_0(\lambda_2 - \lambda_1\delta q_s^2) + \lambda_0\lambda_1 q_X^2) + \lambda_1 q_X^2[2(\lambda_2 - 3\lambda_1\delta q_s^2) + q_X^2] = 0.$$

Since the roots are real and their sum is always negative, the pattern is stable as long as both roots are negative, i.e., their product is positive. The cell pattern becomes unstable when the product is negative, i.e., when

$$q_X^2 \geqslant 2\left(\delta q_s^2 - \frac{\lambda_2}{\lambda_1}\right) \quad \text{and } q_X^2 \leqslant 2(3\lambda_1\delta q_s^2 - \lambda_2),$$

for this requires $\sqrt{\frac{\lambda_2}{3\lambda_1}} \leqslant |\delta q_s| \leqslant \sqrt{\frac{\lambda_2}{\lambda_1}}$; this condition defines the domain of the Eckhaus instability. The above condition implies that the most unstable wave vector tends to zero, when $|\delta q_s| \to \sqrt{\frac{\lambda_2}{\lambda_1}}$.

(b) Transverse perturbations and zigzag instability: Let us consider $q_X = 0$ into Eq. (46), we get

$$\lambda_0^2 S^2 + 2S\left[2\lambda_0(\lambda_2 - \lambda_1\delta q_s^2) + \frac{\lambda_0\lambda_1}{q_{sc}} q_Y^2\delta q_s + \frac{\lambda_0\lambda_1}{4q_{sc}^2} q_Y^4\right] + \left[2(\lambda_2 - \lambda_1\delta q_s^2) + \frac{\lambda_1}{q_{sc}} q_Y^2\delta q_s + \frac{\lambda_1}{4q_{sc}^2} q_Y^4\right]\left[\frac{\delta q_s}{q_{sc}} + \frac{q_Y^2}{4q_{sc}^2}\right]\lambda_1 q_Y^2 = 0.$$

The two eigen modes are uncoupled and we have $S(-)$,

$$S(-) = -2(\lambda_2 - \lambda_1\delta q_s^2) - \frac{\lambda_1}{q_{sc}} q_Y^2\delta q_s - \frac{\lambda_1}{4q_{sc}^2} q_Y^4 < 0,$$

for one of them. The other is amplified when

$$S(+) = -\lambda_1 q_Y^2\left(\delta q_s + \frac{q_Y^2}{4q_{sc}}\right) > 0.$$

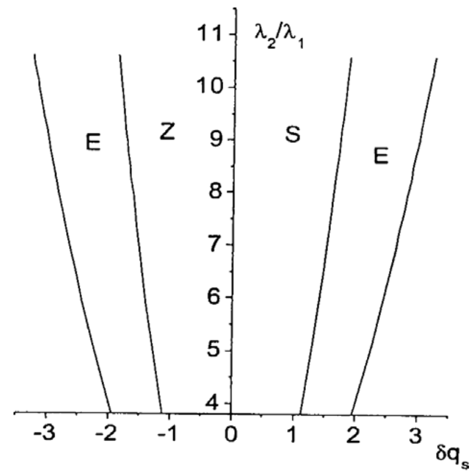

Fig. 3. Eckhuas instability (E) exists between the two lines $\delta q_s^2 = \lambda_2/3\lambda_1$ and $\delta q_s^2 = \lambda_2/\lambda_1$. Zigzag instability (Z) exists in the $\delta q_s < 0$ region. Stable rolls (S) exist in the $\delta q_s > 0$ region of inner curve.

This implies that $\delta q_s < 0$, the above condition defines the domain of the zigzag instability. Since $\lambda_1 > 0$, we get $|\delta q_s| > q_Y^2/4q_{sc}$.

Since from Eq. (37), λ_1 and λ_2 are always positive. Thus the curves $\delta q_s^2 = \lambda_2/3\lambda_1$ and $\delta q_s^2 = \lambda_2/\lambda_1$ will not enter to the center. The regions of Eckhaus instability and zigzag instability increases when Taylor number increases (see Fig. 3).

5. Oscillatory convection at the supercritical Hopf bifurcation

The existence of a threshold (critical value of Rayleigh number for the onset of oscillatory convection $R = R_{oc}$) and a cellular structure (critical wave number $q = q_{oc}$) and Taylor number Ta are main characteristics of the oscillatory convection in the rotating fluid. In this section, we treat region near the onset of oscillatory convection. Here, the axis of the cylindrical rolls is taken as y-axis, so that y-dependence disappears from equation $\mathscr{L}w = \mathscr{N}$. The z-dependence contained entirely in sin and cos functions which ensure that the free–free boundary conditions are satisfied. The purpose of this section is to derive coupled one dimensional nonlinear time dependent Landau–Ginzburg type equations near the onset of oscillatory convection at supercritical Hopf bifurcation. We introduce ϵ as

$$\epsilon^2 = \frac{R - R_{oc}}{R_{oc}} \ll 1. \tag{48}$$

We assume that

$$w_0 = [A_{1L}e^{i(\omega_{oc}t + q_{oc}x)} + A_{1R}e^{i(\omega_{oc}t - q_{oc}x)} + \text{c.c.}]\sin \pi z,$$

is a solution to linearized equation $\mathscr{L}w = 0$, which satisfies free–free boundary conditions. Hear A_{1L} denotes the amplitude of left travelling of the roll and A_{1R} denotes the amplitude of right travelling of the roll, which are dependent on slow space and time variables

$$X = \epsilon x,\ \tau = \epsilon t,\ T = \epsilon^2 t, \tag{49}$$

and assume that $A_{1L} = A_{1L}(X, \tau, T), A_{1R} = A_{1R}(X, \tau, T)$. The differential operators can be expressed as

$$\frac{\partial}{\partial x} \to \frac{\partial}{\partial x} + \epsilon \frac{\partial}{\partial X}, \quad \frac{\partial}{\partial t} \to \frac{\partial}{\partial t} + \epsilon \frac{\partial}{\partial \tau} + \epsilon^2 \frac{\partial}{\partial T}. \tag{50}$$

The solution of basic equations can be sought as power series in ϵ,

$$f = \epsilon f_0 + \epsilon^2 f_1 + \epsilon^3 f_2 + \cdots,$$

where $f = f(u, v, w, \omega_x, \omega_y, \omega_z, \theta)$ with the first approximation is given by eigenvector of the linearized problem:

$$\begin{aligned}
u_0 &= \frac{i\pi}{q_{oc}}\left[A_{1L}e^{i(\omega_{oc}t+q_{oc}x)} - A_{1R}e^{i(\omega_{oc}t+q_{oc}x)} - \text{c.c.}\right]\cos\pi z,\\
v_0 &= \frac{-Ta^{\frac{1}{2}}i\pi}{q_{oc}}\left[\frac{A_{1L}e^{i(\omega_{oc}t+q_{oc}x)}}{\delta_{oc}^2 + \frac{i\omega_{oc}}{Pr}} - \frac{A_{1R}e^{i(\omega_{oc}t-q_{oc}x)}}{\delta_{oc}^2 + \frac{i\omega_{oc}}{Pr}} - \text{c.c.}\right]\cos\pi z,\\
\omega_{x_0} &= \frac{-Ta^{\frac{1}{2}}i\pi^2}{q_{oc}}\left[\frac{A_{1L}e^{i(\omega_{oc}t+q_{oc}x)}}{\delta_{oc}^2 + \frac{i\omega_{oc}}{Pr}} - \frac{A_{1R}e^{i(\omega_{oc}t+q_{oc}x)}}{\delta_{oc}^2 + \frac{i\omega_{oc}}{Pr}} - \text{c.c.}\right]\sin\pi z,\\
\omega_{y_0} &= \frac{-i\delta_{oc}^2}{q_{oc}}\left[A_{1L}e^{i(\omega_{oc}t+q_{oc}x)} - A_{1R}e^{i(\omega_{oc}t+q_{oc}x)} - \text{c.c.}\right]\sin\pi z,\\
\omega_{z_0} &= Ta^{\frac{1}{2}}\pi\left[\frac{A_{1L}e^{i(\omega_{oc}t+q_{oc}x)}}{\delta_{oc}^2 + \frac{i\omega_{oc}}{Pr}} - \frac{A_{1R}e^{i(\omega_{oc}t-q_{oc}x)}}{\delta_{oc}^2 + \frac{i\omega_{oc}}{Pr}} - \text{c.c.}\right]\cos\pi z,\\
\theta_0 &= \left[\frac{A_{1L}e^{i(\omega_{oc}t+q_{oc}x)}}{\delta_{oc}^2 + i\omega_{oc}} + \frac{A_{1R}e^{i(\omega_{oc}t+q_{oc}x)}}{\delta_{oc}^2 + i\omega_{oc}} + \text{c.c.}\right]\sin\pi z,
\end{aligned} \tag{51}$$

where $\delta_{oc}^2 = \pi^2 + q_{oc}^2$. We expand the linear operator $\mathscr{L}$ and nonlinear term $\mathscr{N}$ as the following power series

$$\mathscr{L} = \mathscr{L}_0 + \epsilon\mathscr{L}_1 + \epsilon^2\mathscr{L}_2 + \cdots \tag{52a}$$

$$\mathscr{N} = \epsilon^2\mathscr{N}_0 + \epsilon^3\mathscr{N}_1 + \cdots \tag{52b}$$

Substituting Eqs. (28) and (50) into $\mathscr{L}w = \mathscr{N}$, for each order of ϵ, we get

$$\mathscr{L}_0 w_0 = 0, \tag{53a}$$

$$\mathscr{L}_0 w_1 + \mathscr{L}_1 w_0 = \mathscr{N}_0, \tag{53b}$$

$$\mathscr{L}_0 w_2 + \mathscr{L}_1 w_1 + \mathscr{L}_2 w_0 = \mathscr{N}_1. \tag{53c}$$

Here

$$\begin{aligned}
\mathscr{L}_0 = &\left(\frac{\partial}{\partial t} - \nabla^2\right)\left(\frac{1}{Pr}\frac{\partial}{\partial t} - \nabla^2\right)^2\nabla^2 - R_{oc}\frac{\partial^2}{\partial x^2}\left(\frac{1}{Pr}\frac{\partial}{\partial t} - \nabla^2\right)\\
&+ Ta\frac{\partial^2}{\partial z^2}\left(\frac{\partial}{\partial t} - \nabla^2\right),
\end{aligned} \tag{54a}$$

$$\begin{aligned}
\mathscr{L}_1 = &\frac{\partial}{\partial\tau}\Bigg\{\frac{2}{Pr}\nabla^2\left(\frac{\partial}{\partial t} - \nabla^2\right)\left(\frac{1}{Pr}\frac{\partial}{\partial t} - \nabla^2\right) - \frac{R_{oc}}{Pr}\frac{\partial^2}{\partial x^2}\\
&+ Ta\frac{\partial^2}{\partial z^2} + \nabla^2\left(\frac{1}{Pr}\frac{\partial}{\partial t} - \nabla^2\right)^2\Bigg\}\\
&+ 2\frac{\partial^2}{\partial x\partial X}\Bigg\{\left(\frac{1}{Pr}\frac{\partial}{\partial t} - \nabla^2\right)^2\left(\frac{\partial}{\partial t} - \nabla^2\right)\\
&- 2\left(\frac{1}{Pr}\frac{\partial}{\partial t} - \nabla^2\right)\left(\frac{\partial}{\partial t} - \nabla^2\right)\nabla^2 - \left(\frac{1}{Pr}\frac{\partial}{\partial t} - \nabla^2\right)^2\nabla^2\\
&- Ta\frac{\partial^2}{\partial z^2} - R_{oc}\left(\frac{1}{Pr}\frac{\partial}{\partial t} - \nabla^2\right) + R_{oc}\frac{\partial^2}{\partial x^2}\Bigg\},
\end{aligned} \tag{54b}$$

$$\begin{aligned}
\mathscr{L}_2 = &\frac{\partial}{\partial T}\Bigg[\frac{2}{Pr}\left(\frac{1}{Pr}\frac{\partial}{\partial t} - \nabla^2\right)\left(\frac{\partial}{\partial t} - \nabla^2\right)\nabla^2\\
&+ \left(\frac{1}{Pr}\frac{\partial}{\partial t} - \nabla^2\right)^2\nabla^2 - \frac{R_{oc}}{Pr}\frac{\partial^2}{\partial x^2} + Ta\frac{\partial^2}{\partial z^2}\Bigg]\\
&+ 4\frac{\partial^4}{\partial x^2\partial X^2}\Bigg[2\left(\frac{1}{Pr}\frac{\partial}{\partial t} - \nabla^2\right)\nabla^2 + \nabla^2\left(\frac{\partial}{\partial t} - \nabla^2\right)\\
&- \left(\frac{1}{Pr}\frac{\partial}{\partial t} - \nabla^2\right)^2 - 2\left(\frac{1}{Pr}\frac{\partial}{\partial t} - \nabla^2\right)\left(\frac{\partial}{\partial t} - \nabla^2\right) + R_{oc}\Bigg]\\
&+ \frac{\partial^2}{\partial X^2}\Bigg[\left(\frac{1}{Pr}\frac{\partial}{\partial t} - \nabla^2\right)^2\left(\frac{\partial}{\partial t} - \nabla^2\right) - 2\left(\frac{1}{Pr}\frac{\partial}{\partial t} - \nabla^2\right)\\
&\times\left(\frac{\partial}{\partial t} - \nabla^2\right)\nabla^2 - \left(\frac{1}{Pr}\frac{\partial}{\partial t} - \nabla^2\right)^2\nabla^2\\
&- Ta\frac{\partial^2}{\partial z^2} - R_{oc}\left(\frac{1}{Pr}\frac{\partial}{\partial t} - \nabla^2\right) + R_{oc}\frac{\partial^2}{\partial x^2}\Bigg]\\
&+ 2\frac{\partial^2}{\partial x\partial X\partial\tau}\Bigg[\left(\frac{1}{Pr}\frac{\partial}{\partial t} - \nabla^2\right)^2 + \frac{2}{Pr}\left(\frac{\partial}{\partial t} - \nabla^2\right)\left(\frac{1}{Pr}\frac{\partial}{\partial t} - \nabla^2\right)\\
&- \frac{2}{Pr}\nabla^2\left(\frac{\partial}{\partial t} - \nabla^2\right) - \frac{R_{oc}}{Pr} - 2\left(1 + \frac{1}{Pr}\right)\nabla^2\left(\frac{1}{Pr}\frac{\partial}{\partial t} - \nabla^2\right)\Bigg]\\
&+ \frac{\partial^2}{\partial\tau^2}\nabla^2\left[\frac{1}{Pr^2}\left(\frac{\partial}{\partial t} - \nabla^2\right) + \frac{2}{Pr}\left(\frac{1}{Pr}\frac{\partial}{\partial t} - \nabla^2\right)\right]\\
&- R_{oc}\frac{\partial^2}{\partial x^2}\left(\frac{1}{Pr}\frac{\partial}{\partial t} - \nabla^2\right).
\end{aligned} \tag{54c}$$

Eq. (53a) is a linear problem. We get critical Rayleigh number for the onset of oscillatory convection by using the zeroth order solution w_0 in Eq. (53a). At $O(\epsilon^2), \mathscr{N}_0 = 0$ and $\mathscr{L}w_0 = 0$ gives

$$\frac{\partial A_{1L}}{\partial\tau} - v_g\frac{\partial A_{1L}}{\partial X} = 0 \text{ and } \frac{\partial A_{1R}}{\partial\tau} + v_g\frac{\partial A_{1R}}{\partial X} = 0, \tag{55}$$

where $v_g = (\frac{\partial\omega}{\partial_q})_{q=q_{oc}}$ is the group velocity and is real. Hence from Eq. (53b), we get $w_i = 0$. From equation of continuity we find that $u_1 = 0$. Substituting the zeroth order and first order approximations into Eqs. (34a) and (34b) we get

$$\omega_{z_1} = \frac{Ta^{\frac{1}{2}}\pi^2}{Pr}\left[\frac{A_{1L}^2 e^{2i(\omega_{oc}t+q_{oc}x)}}{\left(\delta_{oc}^2+\frac{i\omega_{oc}}{Pr}\right)\left(2q_{oc}^2+\frac{i\omega_{oc}}{Pr}\right)} + \frac{A_{1R}^2 e^{2i(\omega_{oc}t-q_{oc}x)}}{\left(\delta_{oc}^2+\frac{i\omega_{oc}}{Pr}\right)\left(2q_{oc}^2+\frac{i\omega_{oc}}{Pr}\right)} + \frac{\delta_{oc}^2 A_{1L}A_{1R}^* e^{2iq_{oc}x}}{q_{oc}^2\left(\delta_{oc}^4+\frac{\omega_{oc}^2}{Pr^2}\right)} + \text{c.c.}\right],$$

$$\omega_{x1} = 0, \quad \omega_{y1} = 0,$$

$$\theta_1 = -\pi\left[\frac{\left(|A_{1L}|^2+|A_{1R}|^2\right)\delta_{oc}^2}{2\pi^2\left(\delta_{oc}^4+\omega_{oc}^2\right)} + \frac{A_{1L}A_{1R}e^{2i\omega_{oc}t}}{(2\pi^2+i\omega_{oc})\left(\delta_{oc}^2+i\omega_{oc}\right)} + \text{c.c.}\right],$$

$$v_1 = \frac{-iTa^{\frac{1}{2}}\pi^2}{2Prq_{oc}}\left[\frac{A_{1L}^2 e^{2i(\omega_{oc}t+q_{oc}x)}}{\left(2q_{oc}^2+\frac{i\omega_{oc}}{Pr}\right)\left(\delta_{oc}^2+\frac{i\omega_{oc}}{Pr}\right)} - \frac{A_{1R}^2 e^{2i(\omega_{oc}t-q_{oc}x)}}{\left(2q_{oc}^2+\frac{i\omega_{oc}}{Pr}\right)\left(\delta_{oc}^2+\frac{i\omega_{oc}}{Pr}\right)} + \frac{\delta_{oc}^2 e^{2iq_{oc}x}A_{1L}A_{1R}^* a}{q_{oc}^2\left(\delta_{oc}^4+\frac{\omega_{oc}^2}{Pr^2}\right)} - \text{c.c.}\right], \tag{56}$$

Eq. (53c) is solvable when $\mathscr{L}_0 w_0 = 0$, one requires that its right hand side be orthogonal to w_0, which is ensured that if the coefficients of $\sin \pi z$ in $\mathscr{N}_1 - \mathscr{L}_2 w_0$ are equal to zero. This implies that

$$\Lambda_0\frac{\partial A_{1L}}{\partial T} + \Lambda_1\left(\frac{\partial}{\partial\tau} - v_g\frac{\partial}{\partial X}\right)A_{2L} - \Lambda_2\frac{\partial^2 A_{1L}}{\partial X^2} - \Lambda_3 A_{1L} + \Lambda_4|A_{1L}|^2 A_{1L} + \Lambda_5|A_{1R}|^2 A_{1L} = 0, \tag{57a}$$

$$\Lambda_0\frac{\partial A_{1R}}{\partial T} + \Lambda_1\left(\frac{\partial}{\partial\tau} + v_g\frac{\partial}{\partial X}\right)A_{2R} - \Lambda_2\frac{\partial^2 A_{1R}}{\partial X^2} - \Lambda_3 A_{1R} + \Lambda_4|A_{1R}|^2 A_{1R} + \Lambda_5|A_{1R}|^2 A_{1R} = 0, \tag{57b}$$

where

$$\Lambda_0 = \frac{2\delta_{oc}^2}{Pr}\left(\delta_{oc}^2+i\omega_{oc}\right)\left(\delta_{oc}^2+\frac{i\omega_{oc}}{Pr}\right) + \delta_{oc}^2\left(\delta_{oc}^2+\frac{i\omega_{oc}}{Pr}\right)^2 - \frac{R_{oc}q_{oc}^2}{Pr} + Ta\pi^2,$$

$$\Lambda_1 = \frac{\delta_{oc}^2}{Pr}\left[\frac{1}{Pr}\left(\delta_{oc}^2+i\omega_{oc}\right) + 2\left(\delta_{oc}^2+\frac{i\omega_{oc}}{Pr}\right)\right],$$

$$\begin{aligned}\Lambda_2 = {} & 4q_{oc}^2\left\{2\left(\delta_{oc}^2+i\omega_{oc}\right)\left(\delta_{oc}^2+\frac{i\omega_{oc}}{Pr}\right) + \left(\delta_{oc}^2+\frac{i\omega_{oc}}{Pr}\right)^2 + \delta_{oc}^2\left(\delta_{oc}^2+i\omega_{oc}\right) + 2\delta_{oc}^2\left(\delta_{oc}^2+i\omega_{oc}\right) - R_{oc}\right\} \\ & - v_g^2\frac{\delta_{oc}^2}{Pr}\left[\frac{1}{Pr}\left(\delta_{oc}^2+i\omega_{oc}\right) + 2\left(\delta_{oc}^2+\frac{i\omega_{oc}}{Pr}\right)\right] \\ & + 2iq_{oc}v_g\left[\frac{2}{Pr}\left(\delta_{oc}^2+i\omega_{oc}\right)\left(\delta_{oc}^2+\frac{i\omega_{oc}}{Pr}\right) + \frac{2}{Pr}\delta_{oc}^2\left(\delta_{oc}^2+i\omega_{oc}\right) + \left(\delta_{oc}^2+\frac{i\omega_{oc}}{Pr}\right)^2 + 2\left(1+\frac{1}{Pr}\right)\delta_{oc}^2\left(\delta_{oc}^2+\frac{i\omega_{oc}}{Pr}\right) - \frac{R_{oc}}{Pr}\right] \\ & + \delta_{oc}^2\left(\delta_{oc}^2+\frac{i\omega_{oc}}{Pr}\right)^2 + \left(\delta_{oc}^2+i\omega_{oc}\right)\left(\delta_{oc}^2+\frac{i\omega_{oc}}{Pr}\right)^2 + 2\delta_{oc}^2\left(\delta_{oc}^2+i\omega_{oc}\right)\left(\delta_{oc}^2+\frac{i\omega_{oc}}{Pr}\right) + Ta\pi^2 \\ & - R_{oc}q_{oc}^2 - R_{oc}\left(\delta_{oc}^2+\frac{i\omega_{oc}}{Pr}\right),\end{aligned}$$

$$\Lambda_3 = R_{oc}q_{oc}^2\left(\delta_{oc}^2+\frac{i\omega_{oc}}{Pr}\right),$$

$$\Lambda_4 = R_{oc}q_{oc}^2\pi^2\left(\delta_{oc}^2+\frac{i\omega_{oc}}{Pr}\right)\frac{\delta_{oc}^2}{2\pi^2\left(d_{oc}^4+\omega_{oc}^2\right)} - \frac{2Ta\pi^4\left(\delta_{oc}^2+i\omega_{oc}\right)}{Pr^2\left(\delta_{oc}^2+\frac{i\omega_{oc}}{Pr}\right)\left(4q_{oc}^2+\frac{2i\omega_{oc}}{Pr}\right)},$$

$$\Lambda_5 = R_{oc}q_{oc}^2\pi^2\left(\delta_{oc}^2+\frac{i\omega_{oc}}{Pr}\right)\left[\frac{\delta_{oc}^2}{2\pi^2\left(\delta_{oc}^4+\omega_{oc}^2\right)} + \frac{1}{\left(\delta_{oc}^2+i\omega_{oc}\right)(2\pi^2+i\omega_{oc})}\right] - \frac{Ta\pi^4\delta_{oc}^2}{Pr^2q_{oc}^2\left(\delta_{oc}^2+\frac{\omega_{oc}^2}{Pr^2}\right)}\left(\delta_{oc}^2+i\omega_{oc}\right). \tag{58}$$

It should be noted that A_{1L}, A_{1R} are of order ϵ and A_{2L}, A_{2R} are of order ϵ^2. If $\omega_{oc} = 0$ in $\Lambda_0, \Lambda_2, \Lambda_3$ and Λ_4 then these expressions match with the coefficients $\lambda_0, \lambda_1, \lambda_2$, and λ_3 of Landau–Ginzburg equation at the onset of stationary convection.

5.1. Travelling wave and standing wave convection

To study the stability regions of travelling waves and standing waves we proceed as follows:

On dropping slow space variable X and slow time variable τ from Eqs. (57a) and (57b), we get a pair of first order ODE's

$$\frac{dA_{1L}}{dT} = \frac{\Lambda_3}{\Lambda_0}A_{1L} - \frac{\Lambda_4}{\Lambda_0}A_{1L}|A_{1L}|^2 - \frac{\Lambda_5}{\Lambda_0}A_{1L}|A_{1R}|^2, \tag{59}$$

$$\frac{dA_{1R}}{dT} = \frac{\Lambda_3}{\Lambda_0}A_{1R} - \frac{\Lambda_4}{\Lambda_0}A_{1R}|A_{1R}|^2 - \frac{\Lambda_5}{\Lambda_0}A_{1R}|A_{1L}|^2. \tag{60}$$

Put

$$\beta = \frac{\Lambda_3}{\Lambda_0}, \gamma = -\frac{\Lambda_4}{\Lambda_0} \text{ and } \delta = -\frac{\Lambda_5}{\Lambda_0}.$$

Then Eq. (59) and (60) takes the following form

$$\frac{dA_{1L}}{dT} = \beta A_{1L} + \gamma A_{1L}|A_{1L}|^2 + \delta A_{1L}|A_{1R}|^2, \tag{61}$$

$$\frac{dA_{1R}}{dT} = \beta A_{1R} + \gamma A_{1R}|A_{1R}|^2 + \delta A_{1R}|A_{1L}|^2. \tag{62}$$

Consider $A_{1L} = a_L e^{i\phi L}$ and $A_{1R} = a_R e^{i\phi R}$ (we can write a complex number in the amplitude and phase (angle) form), where $a_L = |A_{1L}|, \phi_L = arg(A_{1L}) = \tan^{-1}\left(\frac{\text{Im}(A_{1L})}{\text{Re}(A_{1L})}\right)$ and $a_R = |A_{1R}|, \phi_R = arg(A_{1R}) = \tan^{-1}\left(\frac{\text{Im}(A_{1R})}{\text{Re}(A_{1R})}\right)$. a_L, a_R, ϕ_L, ϕ_R are functions of time T since A_{1L} and A_{1R} are functions of T. Thus a_L and a_R are positive functions.

Substituting the definitions of A_{1L} and A_{1R} and $\beta = \beta_1 + i\beta_2, \gamma = \gamma_1 + i\gamma_2, \delta = \delta_1 + i\delta_2$ into Eqs. (61) and (62), we get

$$\frac{da_L}{dT} = \beta_1 a_L + \gamma_1 a_L |a_L|^2 + \delta_1 a_L |a_R|^2, \tag{63}$$
$$\frac{d\phi_L}{dT} = \beta_2 + \gamma_2 |a_L|^2 + \delta_2 |a_R|^2, \tag{64}$$
$$\frac{da_R}{dT} = \beta_1 a_R + \gamma_1 a_R |a_R|^2 + \delta_1 a_R |a_L|^2, \tag{65}$$
$$\frac{d\phi_R}{dT} = \beta_2 + \gamma_2 |a_R|^2 + \delta_2 |a_L|^2. \tag{66}$$

Eqs. (63) and (65) not contain phase term, so we take these two equations for the future discussions. We have Eqs. (63) and (65) as

$$\frac{da_L}{dT} = \beta_1 a_L + \gamma_1 a_L^3 + \delta_1 a_L a_R^2,$$
$$\frac{da_R}{dT} = \beta_1 a_R + \gamma_1 a_R^3 + \delta_1 a_R a_L^2,$$

since a_L and a_R are positive functions. Put

$$\frac{da_L}{dT} = F_1(a_L, a_R), \frac{da_R}{dT} = F_2(a_L, a_R) \tag{67}$$

Now we discuss the stability of equilibrium points of above Eq. (67). We get four equilibrium points like $(a_L, a_R) = (0,0)$ [conduction state], $(a_L, a_R) = (a_L, 0)$ [a_L = amplitude of left travelling waves, here we get $F_2 = 0$, and we get one condition from $F_1 = 0$, i.e., $a_L^2 = -\frac{\beta_1}{\gamma_1}(= |A_{1L}|^2)$], $(a_L, a_R) = (0, a_R)$ [a_R = amplitude of right travelling waves, here $F_1 = 0$ and from $F_2 = 0$, we get $a_R^2 = -\frac{\beta_1}{\gamma_1}(= |A_{1R}|^2)$], and for $a_L \neq 0$ and $a_R \neq 0$ we get $(a_L, a_R) = \left(-\frac{\beta_1}{(\gamma_1+\delta_1)}, -\frac{\beta_1}{(\gamma_1+\delta_1)}\right)$ [this gives condition for standing waves. At standing waves we have $A_{1L} = A_{1R}$, so $a_L = a_R$]. For the pair of Eqs. (59) and (60), we do not get $a_L \neq a_R \neq s0$ [modulated waves].

Now the Jacobian of F_1 and F_2 is given by

$$\begin{pmatrix} \frac{\partial F_1}{\partial a_L} & \frac{\partial F_1}{\partial a_R} \\ \frac{\partial F_2}{\partial a_L} & \frac{\partial F_2}{\partial a_R} \end{pmatrix}.$$

If real parts of all eigenvalues of the Jacobian are negative at an equilibrium point, then that point is a stable equilibrium [Lyapounov's theorem or principle of linearized stability]. Some valuable conditions for travelling waves and standing waves are: Travelling waves are stable if $\beta_1 > 0, \gamma_1 < 0$ and $\delta_1 < \gamma_1 < 0$. Standing waves are stable if $\beta_1 > 0, \gamma_1 < 0$ and (i) if $\delta_1 > 0$, then $-\gamma_1 > \delta_1 > 0$, (ii) if $\delta_1 < 0$, then $-\gamma_1 > -\delta_1 > 0$.

The stability regions of travelling waves and standing waves are summarized in Fig. 4. Here E is total amplitude and defined as $E = a_L^2 + a_R^2$. We do not distinguish between left travelling waves and right travelling waves. For rest

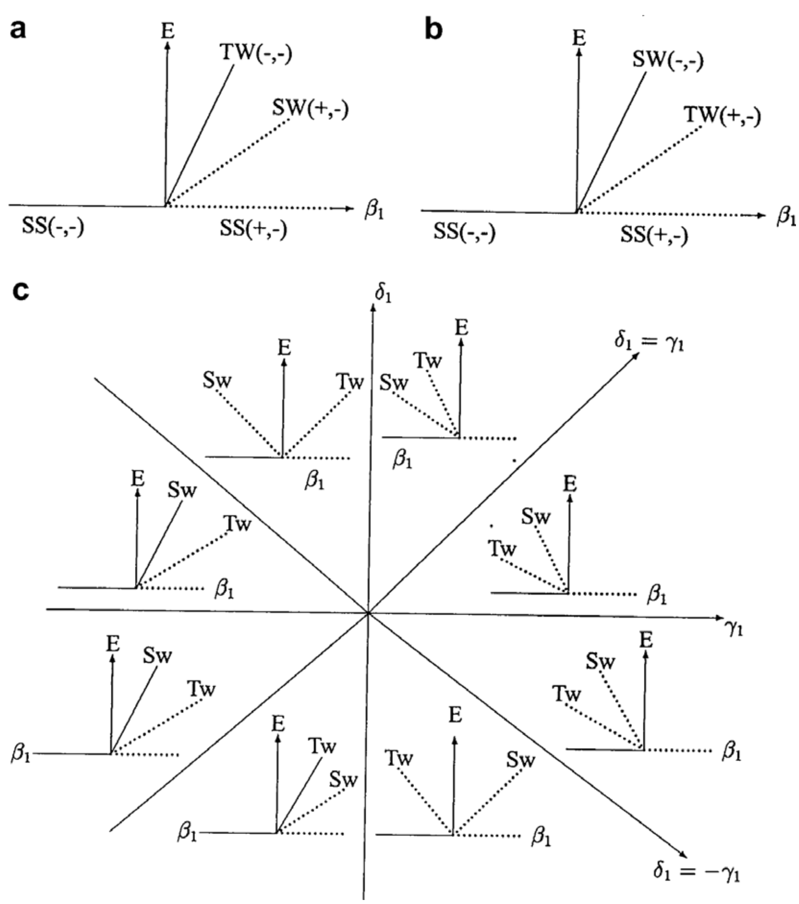

Fig. 4. (a), (b) and (c) are typical diagrams showing the stability of equilibrium solutions SS (steady state), SW (standing waves) and TW (travelling waves). On solid lines equilibrium solutions are stable and on dotted lines they are unstable.

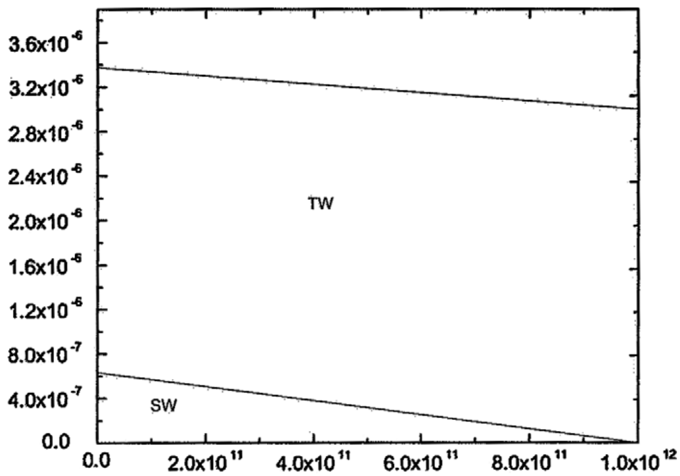

Fig. 5. Above stability diagram is plotted in (Ta, Pr) plane.

state (steady state) $E = 0$, for travelling waves $E = \frac{-\beta_1}{\gamma_1}$, for standing waves $E = \frac{-2\beta_1}{\gamma_1+\zeta_1}$. Travelling waves are supercritical if $\gamma_1 < 0$ and standing waves are supercritical if $\gamma_1 + \zeta_1 < 0$. Fig. 4a is drawn for stable travelling wave conditions and Fig. 4b is drawn for stable standing wave conditions in (β_1, E)-plane. The symbols $(-,-)$ and $(+,-)$ in Fig. 4(a, b) indicate that both two roots of Jacobian are negative and atleast one root is positive among two roots. In Fig. 4(a, b), travelling wave solution and standing wave solution bifurcate simultaneously from the steady state solution ($\beta_1 \geqslant 0$ at this bifurcation point). In these Fig. 4(a, b), steady state solution is stable for $\beta_1 < 0$ and unstable for $\beta_1 > 0$. These figures shows that for $\beta_1 > 0$ both travelling waves and standing waves are supercritical. When travelling waves and standing waves bifurcate supercritically then atmost one solution among travelling waves and standing waves will be stable. Thus, for $\beta_1 > 0$ (Fig. 4a) travelling waves are stable and (Fig. 4b) standing waves are stable. In more detail we reproduce results of the stability analysis of equilibrium solutions in Fig. 4c, which is plotted in (γ_1, ζ_1)-plane. From this figure we can observe that travelling waves are subcritical for $\gamma_1 > 0$ and standing waves are subcritical for $\gamma_1 + \zeta_1 > 0$.

In Fig. 5, we have shown the stability regions for both travelling wave and standing wave for $Pr \ll 1$. It suggest that if travelling waves seen at onset for some values of Pr and Ta, for this fixed Ta as Pr varies then they (travelling waves) loose stability to standing waves soon after the initial bifurcation and vice-versa.

6. Conclusions

We have revisited linear problem of Rayleigh–Benard convection in rotating fluid with the so-called stress-free or free–free boundary conditions and studied nonlinear Rayleigh–Benard convection in rotating fluid near onset of the stationary convection at supercritical pitchfork bifurcation and near onset of oscillatory convection at super critical Hopf bifurcation. Chandrasekhar [4] described the stationary convection and oscillatory convection curves in the R, Ta-plane as curves $R_s(Ta)$ and $R_0(Ta, Pr)$ respectively. In Section 3, we have obtained explicitly Takens–Bogdanov bifurcation point which is the intersection point of the neutral curves corresponds to stationary and oscillatory convection.

In the nonlinear Eq. (36), $\lambda_0 = 0$ gives the condition at the Takens–Bogdanov bifurcation point. The pitchfork bifurcation is supercritical if $\lambda_3 > 0$ and subcritical if $\lambda_3 < 0$. We have computed stability regions of SW and TW at both Hopf bifurcation. The conditions for SW and TW are $A_{1L} = A_{1R}$ and $A_{1L} = 0$ or $A_{1R} = 0$, respectively. TW exist if $|A_{1R}|^2 = -\frac{\beta_1}{\gamma_1} > 0$ and they are supercritical if $\gamma_1 < 0$. SW exist if $|A_{1L}|^2 = |A_{1R}|^2 = -\frac{\beta_1}{\gamma_1+\zeta_1} > 0$ and SW are supercritical if $\gamma_1 + \zeta_1 < 0$. When both SW and TW are supercritical then at most one equilibrium solution is stable. If we substitute $\omega_{oc} = 0$ in the coefficients of Eqs. (57a) and (57b), we get the coefficients of Eq. (36).

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